Subreducts of modules over commutative rings

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Subreducts of modules

Definition

Algebra (A, Ω) is a reduct of a module $(A, +, 0, R)$ if for each $\omega \in \Omega$ there are $r_i^\omega \in R$ such that

$$\omega(x_1, \dots, x_n) = r_1^\omega x_1 + \dots + r_n^\omega x_n.$$

A subreduct is a subalgebra of a reduct.

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Fact

Each subreduct of a module over a commutative ring is entropic, i.e. it satisfies all identities

$$\mu(\nu(x_1^1, \ldots, x_n^1), \ldots, \nu(x_1^m, \ldots, x_n^m)) \approx \nu(\mu(x_1^1, \ldots, x_1^m), \ldots, \mu(x_n^1, \ldots, x_n^m))$$

Cancellative algebras

Cancellation Law

$$\omega(x_1, \dots, y, \dots, x_n) \approx \omega(x_1, \dots, z, \dots, x_n) \longrightarrow y \approx z$$

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Cancellative algebras:

1. (Quasi)Groups,
2. $(R-\{0\},\cdot)$, where R is an integral domain,
3. Let M be a R-module and $r_1,\ldots,r_n\in R-\bigcup_{m\in M}Ann(m)$. If

$$\omega(m_1,\ldots,m_n)=r_1m_1+\ldots+r_nm_n,$$

then the algebra (M,ω) is cancellative.

Polyquasigroups

An algebra (A, Ω) is a polyquasigroup if each translation

$$x \mapsto \omega(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n),$$

where $a_j \in A$ and $\omega \in \Omega$, is bijective.

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Theorem (Sholander, Ježek, Kepka, Stronkowski)

Let $\mathcal{V}$ be a variety of entropic algebras. If an algebra from $\mathcal{V}$ is cancellative, then it is a subalgebra of a polyquasigroup from $\mathcal{V}$.

Romanowska-Smith theorem

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If entropic idempotent algebra (a mode) is cancellative, then it is a subreduct of a module over a commutative ring.

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Proof.

1. Embed a cancellative mode (A, Ω) into a mode polyquasigroup (B, Ω).

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Proof.

1. Embed a cancellative mode (A, Ω) into a mode polyquasigroup (B, Ω).
2. For a basic operation ω of an arity $n > 1$ define

$$\omega_1(x_1, \ldots, x_n) = y \quad \text{iff} \quad \omega(y, x_2, \ldots, x_n) = x_1 \qquad \text{and}$$
$$\omega_n(x_1, \ldots, x_n) = y \quad \text{iff} \quad \omega(x_1, x_2, \ldots, y) = x_n$$

□

proof, continued.

3. Then the operation

$$M(x,y,z) = \omega(\omega_1(x,z,\ldots,z), y, \ldots, y, \omega_n(\omega_1(y,z,\ldots,z), y, \ldots, y, z))$$

is Mal'cev and (B, Ω, M) is a Mal'cev mode equivalent to an affine space.

□

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In fact the same proof gives us

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Theorem

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And

Theorem

Each entropic cancellative algebra is quasi-affine.

We will need the following Szendrei identities

$$\omega(\omega(x_1^1, \dots, x_n^1), \dots, \omega(x_1^n, \dots, x_n^n)) \\ \approx \omega(\omega(\pi(x_1^1), \dots, \pi(x_n^1)), \dots, \omega(\pi(x_1^n), \dots, \pi(x_n^n))),$$

where π is a transposition of a pair of variables x_j^i and x_i^j.

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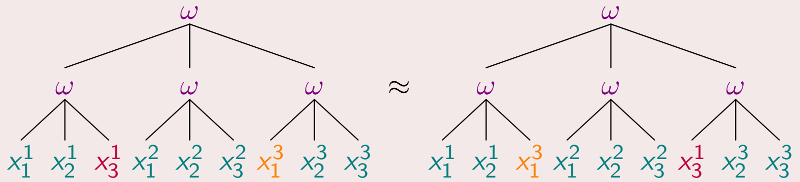
where π is a transposition of a pair of variables x^i_j and x^j_i.

Example: $\pi: x^1_3 \leftrightarrow x^3_1$



Approach through semimodules

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Theorem (Ježek, Kepka, Stronkowski)

Let A be an entropic algebra without constants, satisfying all Szendrei identities and such that each its basic operation of arity at least 2 is onto. Then A is a subreduct of a semimodule over a commutative semiring.

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Remark

Entropic polyquasigroups without constants satisfy assumptions of the previous theorem.

Approach through semimodules, continued

Main Theorem

Each cancellative entropic algebra is a subreduct of a module over a commutative ring.

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Proof.

Let A be cancellative and entropic

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Let A be cancellative and entropic

1. Case when A has constants falls into earlier Theorem, so we assume that A does not have any constant

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Each cancellative entropic algebra is a subreduct of a module over a commutative ring.

Proof.

Let A be cancellative and entropic

1. Case when A has constants falls into earlier Theorem, so we assume that A does not have any constant
2. We embed A into an entropic polyquasigroup B

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3. Next embed B into a semimodule N over a commutative semiring

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2. We embed A into an entropic polyquasigroup B
3. Next embed B into a semimodule N over a commutative semiring
4. One may prove that, by cancellativity of B, the semimodule N is $+$-cancellative and thus

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3. Next embed B into a semimodule N over a commutative semiring
4. One may prove that, by cancellativity of B, the semimodule N is $+$-cancellative and thus
5. embeds into a module over a commutative ring

□

The End

Thank you for your attention :-)